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Ivković's N-Dimensional Chaotic Attractors: A Novel Framework for Multi-Dimensional Dynamical Systems

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Abstract:

This paper introduces a new class of higher-order chaotic dynamical systems, named after the author as Ivković's N-dimensional attractor systems, which generalize the inherited chaotic-fractal properties of their subsystems - the building blocks. These systems utilize multi-dimensional dynamic generators of chaotic attractors, such as independent asynchronous Lorenz or Rössler attractors, to generate complex dynamics and bistable states in multi-dimensional spaces. The paper provides a detailed mathematical model, theoretical analysis, and numerical simulations. The idea for this new approach stemmed from exploring the application possibilities of dynamic systems in artificial intelligence, quantum physics, and economic dynamics. The paper also considers and introduces the practical application of implementing an analog computer for real-time physical verification of these models. The results show that the concept of N-dimensional grouped chaotic attractors proposed by the author offers a robust solution for modeling complex dynamic systems with multiple axes of attraction or expected states, which, due to the nature and conditions of the environment, are never absolutely exact, opening new possibilities for research and application of dynamic chaotic systems with high levels of dimensionality.

Keywords: N-dimensional attractors, chaotic systems, Lorenz attractors, Rössler attractors, artificial intelligence, quantum physics, economic dynamics, bistable states, numerical simulation, analog computers.

1. Introduction

Chaotic systems are fundamental for understanding complex dynamic phenomena in nature. Lorenz [1] and Rössler attractors [2] are among the most well-known examples of such systems. These attractors are crucial for studying the dynamics of fluid systems, meteorological patterns, electronic circuits, and many other phenomena [1, 2]. Despite significant advancements in researching these systems, there is a need to develop new models that can encompass more complex dynamic phenomena in higher dimensions.

This paper presents a new class of chaotic systems known as Ivković's N-dimensional attractors. These attractors extend the concept of classical attractors to higher dimensions, enabling the modeling of complex multi-dimensional dynamic systems. Each attractor is treated as an independent generator of chaotic behavior, and the interactions between them create complex dynamic patterns in multi-dimensional spaces.

The goal of this research is to provide a detailed mathematical model, theoretical analysis, and numerical simulations of these attractors, as well as to explore their potential applications in various scientific disciplines, including artificial intelligence, quantum physics, and economic dynamics. Additionally, we explore the practical implementation of an analog computer for the physical verification of these models, providing a robust framework for future research in the field of multi-dimensional chaotic systems.

2. Fundamentals of Chaotic Systems

Chaotic systems are deterministic by nature but highly sensitive to initial conditions, meaning that small changes in initial conditions can lead to significantly different outcomes. This behavior is often described through Lyapunov exponents, which quantify the rate at which nearby initial conditions diverge. Classic examples of chaotic systems include the Lorenz attractor and the Rössler attractor [3, 4], and they will serve as the fundamental building blocks for our N-System.

Lorenz Attractor:

$\frac{dx}{dt} = \sigma(y - x),$ $\frac{dy}{dt} = x(\rho - z) - y,$	(1)

$\frac{dz}{dt} = xy - \beta z,$	

where σ, ρ, β are system parameters [5].

Rössler Attractor:

$\frac{dx}{dt} = -y - z,$ $\frac{dy}{dt} = x + ay,$ $\frac{dz}{dt} = b + z(x - c),$	(2)

where$a,\ b, c$ are system parameters [6].

Proposal for defining the class of N-Attractors

The author's initial idea is that for any integer N, a class of N-attractors can be generated, representing chaotic and bistable states with two points of attraction. These points are values that the systems tend towards and are derived from a set of their (usually three) variable values. For each N, this class can be viewed as a transformation of a more complex multi-dimensional set of variables, where the number of variables S is grouped into sets of three dimensions {a,b,c}{a,b,c}, which are independent of other dimensions {α,β,γ} in some other set N+1

Ivković's N-dimensional attractors extend the concept of classical attractors to higher dimensions. Each attractor chaotic system is treated as an independent generator of chaotic behavior, and the interactions between them create complex multi-dimensional dynamics with the propagation of key inherited properties upwards to the higher-dimensional meta-N-system. The fundamental building block can be any chaotic-fractal dynamic system with attractor points.

Key questions in proving the existence of such a meta-System:

The key question in proving the possibility of the existence of such a structured meta-system is related to the basic properties of dynamic chaotic-fractal systems, the building elements, and their retention at a higher meta-level. Therefore, proving the possibility of infinitely long trajectories in a bounded space is the starting point. To prove that the trajectory of a point in our system is an infinitely long curve in bounded space, we can use mathematical principles related to fractals and chaotic systems. The key aspects are:

- **Definition of Chaos:** Chaotic systems have the property of sensitivity to initial conditions, meaning that small changes in initial conditions will lead to significantly different trajectories. This property is called sensitivity to initial conditions.
- **Fractal Dimension:** Lorenz attractors and similar systems are known for their fractal structure. Fractal attractors have a dimension that is not an integer and show self-similarity at different scales.
- **Infinite Trajectories:** In chaotic systems, the trajectory never repeats itself and never returns to the exact same point. This can be proven using Lyapunov exponents, which quantify the divergence of nearby trajectories [7].

3. Mathematical model and description of a generic N-Attractor:

The general approach to generating our N-generator can be mathematically represented as a set of independent differential equations modeling chaotic attractors. Each attractor is defined with its parameters and initial conditions, but all attractors operate independently of each other. Mathematically, this system can be described as follows:

$\frac{dx_i}{dt} = f_i(x_i, y_i, z_i),\ \frac{dy_i}{dt} = g_i(x_i, y_i, z_i),\ \frac{dz_i}{dt} = h_i(x_i, y_i, z_i),$	(3)

where i denotes the indices of the attractors, $i = 1, 2, ..., N,\ \ and\ i,\ N \in N\ is\ the\ set\ of\ natural\ numbers,\ while\ f_i,\ g_i,\ h_i$ are functions that define the dynamic behavior of each attractor This generator is considered a set of variables x, y, z in 3D space, dynamically changing over time t.

Transformation for describing the 3N-Dimensional Space:

$X = \{x_1, x_2, ..., x_N\},\ Y = \{y_1, y_2, ..., y_N\},\ Z = \{z_1, z_2, ..., z_N\}$	(4)

This generalization allows modeling complex multi-dimensional systems that can be used to analyze various phenomena such as quantum states, artificial intelligence, and complex dynamic systems. Its building blocks are already known dynamic chaotic-fractal attractor generators for N independent generators, such as:

The proposed (Ivković's) N-dimensional chaotic system composed of multiple Lorenz attractors can be represented by the following differential equations:

$$\begin{aligned}
&\frac{dx_1}{dt} = \sigma(y_1 - x_1),\ \frac{dy_1}{dt} = x_1(\rho - z_1) - y_1,\ \frac{dz_1}{dt} = x_1 y_1 - \beta z_1,\\
&\frac{dx_2}{dt} = \sigma(y_2 - x_2),\ \frac{dy_2}{dt} = x_2(\rho - z_2) - y_2,\ \frac{dz_2}{dt} = x_2 y_2 - \beta z_2,\\
&\qquad\qquad \cdots\\
&\frac{dx_N}{dt} = \sigma(y_N - x_N),\ \frac{dy_N}{dt} = x_N(\rho - z_N) - y_N,\ \frac{dz_N}{dt} = x_N y_N - \beta z_N,
\end{aligned} \qquad (5)$$

For Rössler attractors, the system can be represented as follows:

$$\begin{aligned}
&\frac{dx_1}{dt} = -y_1 - z_1,\ \frac{dy_1}{dt} = x_1 + a y_1,\ \frac{dz_1}{dt} = b + z_1(x_1 - c),\\
&\frac{dx_2}{dt} = -y_2 - z_2,\ \frac{dy_2}{dt} = x_2 + a y_2,\ \frac{dz_2}{dt} = b + z_2(x_2 - c),\\
&\qquad\qquad \cdots\\
&\frac{dx_N}{dt} = -y_N - z_N,\ \frac{dy_N}{dt} = x_N + a y_N,\ \frac{dz_N}{dt} = b + z_N(x_N - c),
\end{aligned} \qquad (6)$$

Lyapunov exponents:

Lyapunov exponents quantify the divergence of nearby trajectories:

$$\lambda = \frac{1}{t} \ln ln \left(\frac{d(t)}{d(0)}\right), \qquad (7)$$

where $d(t)$ represents the distance between two infinitely close trajectories over time t. A positive Lyapunov exponent indicates chaotic behavior, i.e., exponential growth of initial differences, which is satisfied in both previous models [8]. This value changes with each evolution, i.e., trajectory cycle, and is calculated based on simulations.

Fractal dimension:

The fractal dimension D can be calculated as:

$$D = j + \frac{\sum_{i=1}^{j} \lambda_i}{|\lambda_{j+1}|} \qquad (8)$$

where λ_i are the Lyapunov exponents, and j is the largest integer for which the condition is met: $\sum_{i=1}^{j} \lambda_i \geq 0$. The fractal dimension is a decimal number, indicating the complexity and infinity of the attractor's structure [9]. The number is calculated numerically based on the sum of successively measured exponents, i.e., based on the divergence of trajectories countable with $ij = 1, 2, ..., n,$ and λ_{min} is the smallest Lyapunov exponent.

Proof of infinite trajectories for an Arbitrary Point

We will prove that the path of an arbitrary curve through points $A(x1, x2, x3)$ formed as derivatives of positions $x1,\ x2,\ x3$ of each of the three independent and asynchronous (N=3) chaotic attractor generators on X in 3D space also inherits the properties of each of them. Thus, $A(x,\ y,\ z) = \left\{\frac{dx_1}{dt}, \frac{dx_2}{dt}, \frac{dx_3}{dt}\right\}$ from an N-dimensional chaotic system never passes through the same point twice.

3.1. This proof involves the following steps:

1. Divergence Analysis: Using Lyapunov exponents to quantify the divergence of nearby trajectories.

Lyapunov exponents are crucial for understanding the divergent properties of chaotic systems. A positive Lyapunov exponent indicates exponential divergence of nearby trajectories. As seen in the previous equations (7) and (8), where the fractal dimension is calculated for each $d(t)$, representing the distance, i.e., perturbation between two close points on the trajectory over time t. Thus, according to equation (8), the fractal dimension of chaotic attractors can be expressed through Lyapunov exponents. Assuming the Lyapunov exponents of the system are $\lambda_1, \lambda_2, \ldots, \lambda_n$

Lyapunov exponents are used to demonstrate stability. Lyapunov's exponents determine how perturbations grow or decline in time. For chaotic systems, one or more Lyapunov exponents is positive, indicating sensitivity to initial conditions.

2. Fractal Structure of the Attractor: Analyzing the fractal dimension showing that the attractor occupies an infinite number of points in a finite space.

The fractal dimension of a chaotic attractor provides a measure of the complexity and infinity of its structure. Attractors with a fractal dimension have an infinite number of points in finite space.

Kaplan-Yorke Dimension:

$$D_{KY} = K + \frac{\sum_{j=1}^{N} \lambda_j}{|\lambda_{K+1}|} \quad (9)$$

is a case of formula (8) where λ_j are the Lyapunov exponents, and K is the largest integer for which the condition is met: $\sum_{i=1}^{N} \lambda_i \geq 0$. *tj. najveći indeks za koji je suma eksponenata pozitivna.*

3. Aperiodicity and Infinite Trajectories:

Chaotic systems have positive Lyapunov exponents, meaning their paths are sensitive to initial conditions and never repeat. Trajectories in fractal space are infinitely long and never return to the same point.

Checking Path Overlap:

$$IF \left| X_i(t) - X_j(t) \right| < \varepsilon, \; FOR\, i \neq j, \; THEN\, collision == True. \quad (10)$$

4. Mathematical Representation:

For initial conditions $X_{0'}$, $Y_{0'}$, i $Z_{0'}$, the path of point A(t) can be represented as a series of points $(x(t), y(t), z(t))$. Proving non-overlapping is based on the fact that:

$$\|A(t_1) - A(t_2)\| \neq 0 \quad \forall t_1, t_2 \in R, \; t_1 \neq t_2 \quad (11)$$

This approach demonstrates how chaotic systems with independent generators retain their complex structure and infinite trajectories while allowing for numerical analysis of overlaps in multi-dimensional spaces.

5. Numerical Simulations:

Numerical simulations of chaotic systems enable the visualization of trajectories in multi-dimensional space. As an example, we implemented independent and asynchronous Lorenz attractors in Python, checking for path intersections and marking those points with red dots on the graph. The entire code runs with 1M cycles (periods of oscillation).

Program code:

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import solve_ivp
def lorenz(t, state, sigma, beta, rho):
    x, y, z = state
    dxdt = sigma * (y - x)
    dydt = x * (rho - z) - y
    dzdt = x * y - beta * z
    return [dxdt, dydt, dzdt]
# Parametri Lorencovog sistema
sigma = 10.0
beta = 8.0 / 3.0
rho = 28.0
brciklusa = 1000000
# Početni uslovi za tri nezavisna atraktora
initial_conditions = [
    [1.0, 1.0, 1.0],
    [1.1, 1.1, 1.1],
    [1.2, 1.2, 1.2]]
# Fazni pomaci za asinhrone atraktore
phase_shifts = [0, np.pi / 6.1111, np.pi / 3.1415]
# Vremenski opseg
t_span = (0, 50)
t_eval = np.linspace(t_span[0], t_span[1], brciklusa)
# Funkcija za generisanje podataka sa faznim pomakom
def generate_data(initial_condition, phase_shift):
        solution = solve_ivp(lorenz, t_span, initial_condition, args=(sigma, beta, rho),
t_eval=t_eval)
    phase = np.linspace(0, 2 * np.pi, len(t_eval)) + phase_shift
    x = solution.y[0] * np.cos(phase)
    y = solution.y[1] * np.sin(phase)
    z = solution.y[2]
    return x, y, z
# Generisanje podataka za tri atraktora
```

```
data = [generate_data(init_cond, phase_shift) for init_cond, phase_shift in
zip(initial_conditions, phase_shifts)]
# Crtanje 3D trajektorije sa proverkama prolaza kroz tačke
fig = plt.figure()
ax = fig.add_subplot(111, projection='3d')
# Promenljive za detekciju prelaza kroz tačke
visited_points = set()
collision_count = 0
# Boje i oznake za atraktore
colors = ['r', 'g', 'b']
labels = ['Atraktor 1', 'Atraktor 2', 'Atraktor 3']
for (x, y, z), color, label in zip(data, colors, labels):
    ax.plot(x, y, z, color=color, label=label, linewidth=0.2 )
    for i in range(len(x)):
        point = (round(x[i], 5), round(y[i], 5), round(z[i], 5))
        if point in visited_points:
            ax.scatter(x[i], y[i], z[i], color='red', s=10)
            collision_count += 1
        visited_points.add(point)
...
```

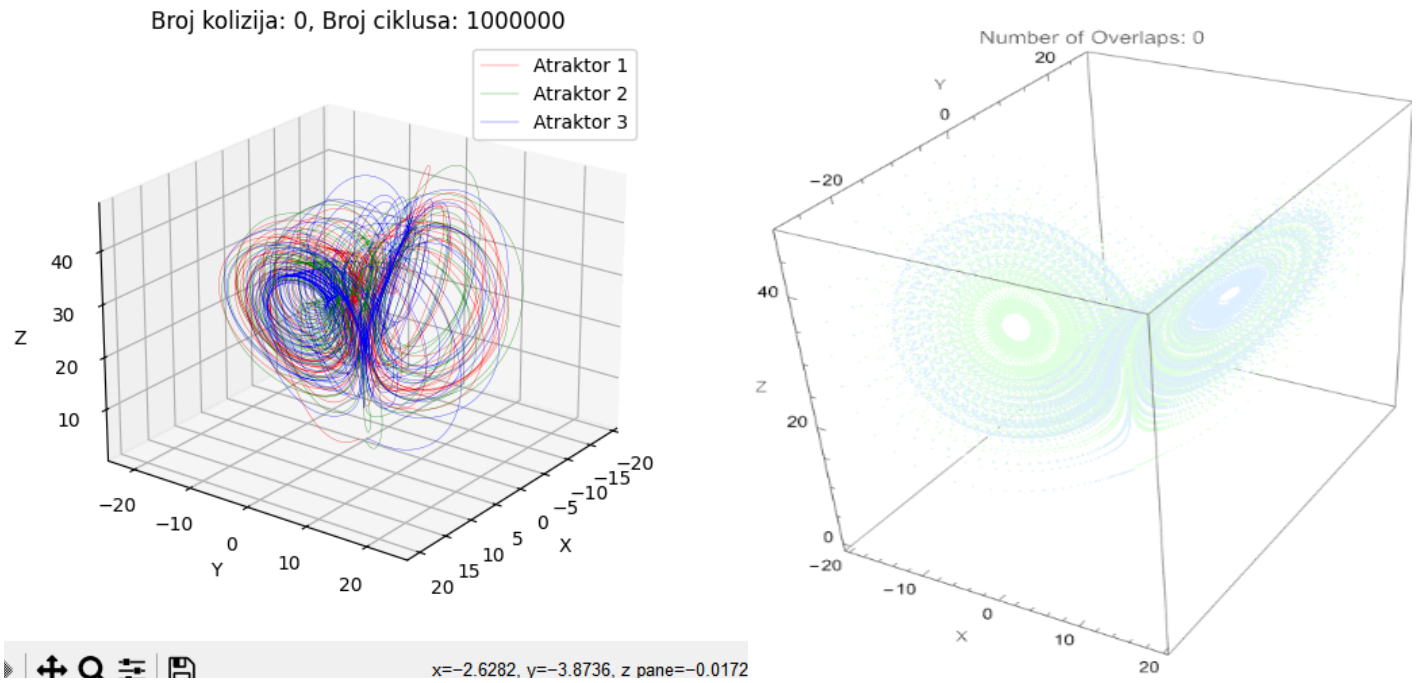


Figure 1. Appearance of simulated: 3N-system with 3 asynchronous (left) Lorenz generators, 2N-system with 2 synchronous Lz. Gen (right) in 3D space

4. Detailed Mathematical Explanation and Proof of Infinite Non-Overlapping Trajectories

To mathematically analyze the possibility of overlapping paths independently generated by different chaotic generators, we need to apply the following approach:

1. **Defining the State of the Generators:** Suppose we have N chaotic generators. Each generator G_i(for $i = 1, 2,\ldots,N$) produces a path defined by the coordinates $\left(x_i(t), y_i(t), z_i(t)\right)$ in three-dimensional space. The path is described by a system of differential equations dependent on initial conditions and system parameters. For the Lorenz system based on (1) and (5), we generalize:

$$\frac{dx_i}{dt} = \sigma\left(y_i - x_i\right),\ \frac{dy_i}{dt} = x_i\left(\rho - z_i\right) - y_i,\ \frac{dz_i}{dt} = x_i y_i - \beta z_i, \qquad (12)$$

2. **Discretizing the Path**: To numerically analyze the paths, we discretize the continuous paths into M time steps. Thus, for each generator G_i we have a series of points $\left(x_i(k), y_i(k), z_i(k)\right)$ for each time step $k\ where\ k = 1, 2, \ldots, M$

3. **Defining the Distance**: To determine if two points overlap, we use a measure of distance. The distance between two points A $\left(x_i(k), y_i(k), z_i(k)\right)$ and B $\left(x_j(l), y_j(l), z_j(l)\right)$ is defined as the Euclidean distance:

$$\overline{AB} = d_{ijkl} = \sqrt{\left(x_i(k) - x_j(l)\right)^2 + \left(y_i(k) - y_j(l)\right)^2 + \left(z_i(k) - z_j(l)\right)^2} \quad (13)$$

If the distance $d_{ijkl} \leq \varepsilon$ for some small value ε, we consider the paths to overlap. This check is limited by the applied precision and resolution of the computation.

4. **Checking for Overlap:**

We check for overlap for all time steps and all pairs of points:

$$C = \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \sum_{k=1}^{M} \sum_{l=1}^{M} H\left(\varepsilon - d_{ijkl}\right) \quad (14)$$

where H is the Heaviside function that has a value of 1 if its argument is positive or zero, and 0 if negative.

4.1. Proof of Infinite Trajectories

Lyapunov Exponents: Lyapunov exponents $\lambda_1, \lambda_2, \ldots, \lambda_n$ determine the divergence of paths in a chaotic system. Positive Lyapunov exponents imply chaos and exponential divergence of paths. In our specific case, each chaotic attractor generator in N=1 dimension is a Lorenz system with chaotic behavior, i.e., with parameter values (σ = 10, β = 8/3, ϱ = 28), the Lyapunov exponents are approximately: $\lambda 1 \approx 0.9056$, $\lambda 2 \approx 0$, $\lambda 3 \approx -14.572$. A positive exponent indicates chaos [10].

λ_1 and all other$\lambda_{i,1}$ are positive, indicating chaos.

λ_2 and all other $\lambda_{i,2} \approx 0$, indicating marginal stability.

λ_3 and all other $\lambda_{i,3}$ are negative, indicating contraction in a specific direction.

Kaplan-Yorke Dimension The fractal dimension according to equation (8) is calculated as:

$$D_{KY} = K + \frac{\sum_{j=1}^{N} \lambda_j}{\left|\lambda_{K+1}\right|} \quad (15)$$

where k is the largest integer for which $\sum_{i=1}^{N} \lambda_i \geq 0$. In our specific case, each chaotic attractor generator in N=1 dimension is a Lorenz system with:: $\lambda 1 \approx 0.9056$, $\lambda 2 \approx 0$, $\lambda 3 \approx -14.572$ The fractal dimension of our simulated Lorenz attractor is:

$$D_{KY} = 2 + \frac{\lambda_1 + \lambda_2}{\left|\lambda_{3j}\right|} \approx 2 + \frac{0.9056}{14.572} \approx 2.0622 \quad (16)$$

In our newly introduced N-dimensional attractor system with 3N systems, i.e., with 3N Lorenz attractors generating a path curve based on only X components (x1,x2, X3) we can estimate:

$$D_{total} = 3N \times D_{Lorenz} \quad (17)$$

where D_{Lorenz} is the fractal dimension of one Lorenz attractor system. In our case for N=1, it is around 2.0622, and since our system has 3 systems operating independently and asynchronously:

$$D_{total} = 3 \times 2.0622 \approx 6.1866 \quad (18)$$

The highest dimension of the system would be when the number of attractor systems N increases, meaning the number of dimensions is directly proportional to the number of attractor systems - generators in the N-system. We can conclude that this method can linearly scale fractal dimensionality, implying that such realized systems based on our concept can model very complex and dynamic processes in high-dimensional spaces.

4.2. Fractal Nature: The paths in chaotic systems are fractal and never repeat, meaning there are no points t1 and t2 for which ‖A(t1) - A(t2)‖=0 except for the trivial case t1 = t2. The basic postulates of proving the possibility of combining trajectories in the N-system while maintaining their chaotic nature are therefore based on the following:

- **Fractal Nature:** Each Lorenz attractor has a fractal nature, meaning it has an infinite number of points and never repeats.
- **Independence and Asynchrony:** Two or more independent systems retain chaos even when combined.
- **Combined Trajectory:** Combining x1 and x2 with z1, (as in the case of x1,x2,x3 for N=3) produces a new trajectory that retains the fractal nature if the initial systems are in a chaotic-fractal state.

4.3. Boundary Conditions:

- **Initial Conditions:** Different initial conditions for each generator ensure different trajectories, minimizing the possibility of overlap.

- **System Parameters:** All generators use the same parameters (σ,β,ϱ) to maintain consistency in behavior, but asynchrony is introduced through different initial states and phase shifts.

5. Simulation and Visualization (Discussion)

Through testing, we observed the high time demands of the model for simulation. Besides the typically slow Python interpreter, the problem also manifested on faster platforms such as Mathematica, Julia, and C++.
To generate the tested Lorenz system, the following is required:

- **Evaluation of the Lorenz Function**: Each evaluation requires 7 floating-point operations, and the function is called 4 times per step (Runge-Kutta fourth-order method).
- **Total Number of Operations per Step**: 4 x 7 = 28.
- **Total Number of Steps**: tmax/dt = 1000 cycles / 0.001 step in a cycle = 1M steps.,
- **Total Number of Operations** for Simulating 1000 Cycles (1kHz) of the Dynamic System: 28 x 1M = 28 MFLOPS FP64, which again is not the whole picture as the code performs a series of operations creating temporary data storage and so on.

If we generate 2N or 3N systems, the number of required operations increases linearly. Interestingly, on a modern CPU with 8 cores (16 threads) and a clock speed of over 4.5GHz, the compiled and optimized C++ code for collision detection executed a maximum of about 800 steps per second, i.e., 22400 OPS, which opens the possibility of considering other non-digital computing options.

C++ Code for path overlap check -part:

```
// Check for overlapping trajectories
int check_overlap(const vector<vector<double>>& data, double epsilon) {
    int count = 0;
    int n = data.size();

#pragma omp parallel for reduction(+:count)  // for Paralelno computation
    for (int i = 0; i < n; ++i) {
        for (int j = i + 1; j < n; ++j) {
            double dist = sqrt(pow(data[i][0] - data[j][0], 2) +
                pow(data[i][1] - data[j][1], 2) +
                pow(data[i][2] - data[j][2], 2));
            if (dist <= epsilon) {
                count++;
            }
        }
    }
    return count;
} //
```

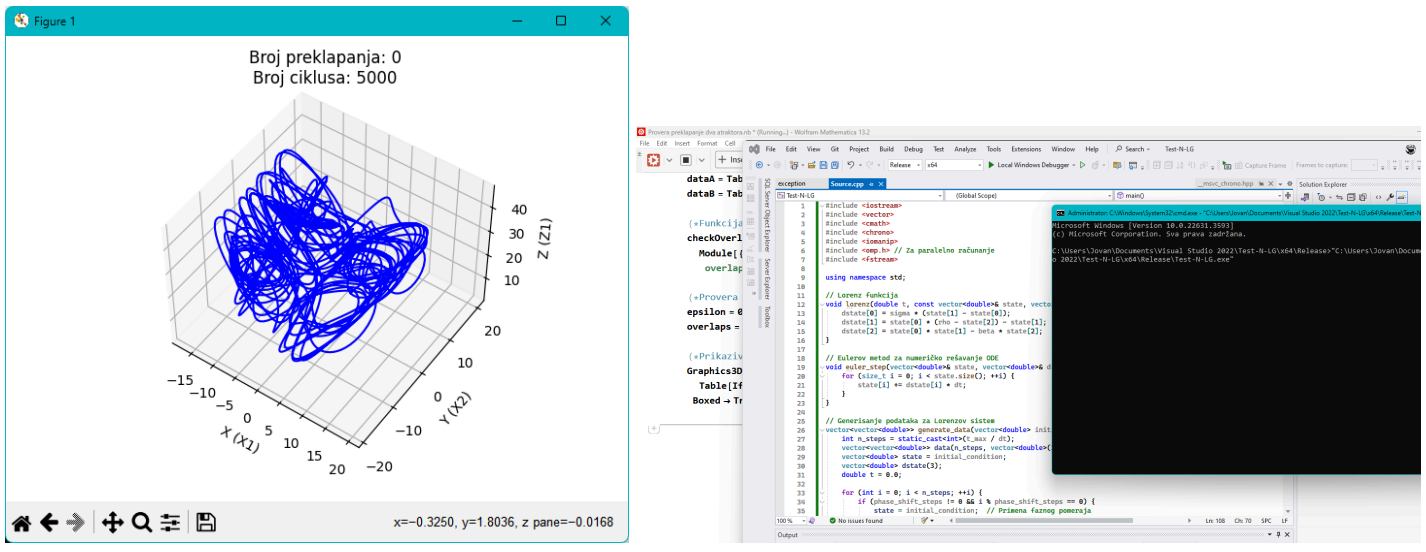


Figure 2. Simulated overlap check function for 2N-systems in Python (left), Simulation running in MS Visual C++ (right)

5.1. Simulation with Electronic Circuits and Real-time Evaluation

For the purpose of evaluating this model in real-time, we developed an electronic circuit based on operational amplifiers (Op.Amp) and multipliers/dividers. In this specific case, it is an application-specific analog computer for simulating 2N and

3N-systems. The fact that these circuits easily generate frequency periods in the range of 20kHz to 1MHz allowed us to test the models in real-time, which on a computer took tens of minutes or hours, and observe their transitions from stable adaptive to chaotic-fractal forms.

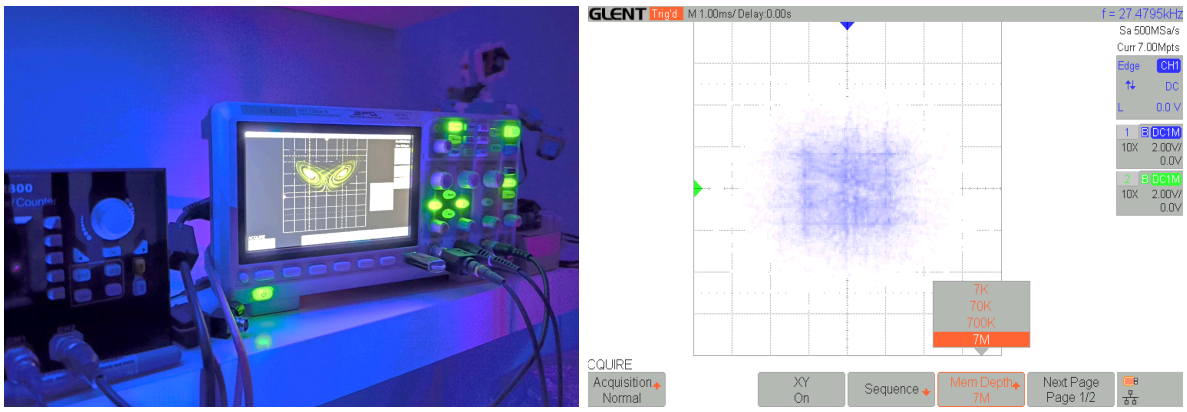


Figure 2 Simulation of 2N chaotic trajectories in real-time on an analog computer displayed on an oscilloscope (left). The appearance of chaotic trajectories developing into butterfly shapes as a result of the work of the analog electronic circuit on a 1N-system (Lorenz) (right).

In Figure 4, we see the distribution that confirms the existence of 2N chaotic trajectories in real-time in the X-Y plane, similar to those simulated in Figures 1-3. Figure 3 shows a detailed view of chaotic trajectories developing into butterfly shapes as a result of the analog circuit's operation. The possibility of developing and applying analog computers with a tolerance of less than 1%, i.e., 0.2%, could be interesting for further study and research of these phenomena.

5. Conclusions

This research introduces an innovative approach to generating and analyzing multi-dimensional chaotic systems through the concept of Ivković's N-dimensional attractors. The presented models enable the generation of multi-dimensional fractal maps that accurately represent the dynamics of N-attractors, providing a tool for visualizing and analyzing complex behavior patterns in multi-dimensional spaces. This technique has proven useful in exploring applications in artificial intelligence, quantum physics, and economic dynamics, opening new possibilities for improving machine learning algorithms, simulating quantum states, and modeling complex financial systems.

The research results show that N-dimensional chaotic attractors provide robust solutions for modeling complex dynamic systems with high levels of dimensionality. The fractal structure and infinite trajectories of these systems provide new tools for analyzing and understanding chaotic phenomena, which can have wide applications in various scientific disciplines. Our work contributes to the development of chaos theory by providing a new framework for future research and practical applications in artificial intelligence, quantum physics, and economic dynamics.

Furthermore, the implementation of an analog computer for the physical verification of these models has proven successful, enabling real-time evaluation and validation of complex dynamic systems. This research emphasizes the importance of an interdisciplinary approach in studying chaotic systems and suggests potential directions for future research, including the development of advanced analog and digital computing systems for simulating and analyzing multi-dimensional chaotic phenomena [1, 2, 3, 4, 5, 6].

Practical applications of this model with N-attractors are possible in areas such as artificial intelligence, quantum physics, economics, and more. The idea for developing this model, solutions, and conceptualizing this field stemmed from numerous works dealing with the practical applicability of complex dynamic systems for describing neuronal functions in neurosynaptic models in the field of artificial intelligence, among others [7,..,24].

Our work has also implemented an analog computer to verify the validity of these models, and its application areas could be:

- **Artificial Intelligence:** These models can help in understanding complex patterns in data and improving machine learning algorithms.
- **Quantum Physics:** Simulation of quantum states and exploration of quantum chaos.
- **Economic Models:** Modeling complex financial systems and markets that exhibit chaotic behavior.

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